Inverse meson mass ordering in color-flavor-locking phase of high density QCD

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Abstract

We derive the effective Lagrangian for the low-energy massive meson excitations of the color-flavor-locking (CFL) phase of QCD with 3 flavors of light quarks. We compute the decay constants, the maximum velocities, and the masses of the mesons at large baryon chemical potential μ. The decay constants are linear in μ. The meson maximum velocities are close to that of sound. The meson masses in the CFL phase are significantly smaller than in the normal QCD vacuum and depend only on bare quark masses. The order of the meson masses is, to some extent, reversed compared to that in the QCD vacuum. In particular, the lightest particle is η'.

1 Introduction

The behavior of QCD at finite baryon density could affect the physics of neutron stars, supernovas and of heavy-ion collisions. Quark pairing at the Fermi surface leading to diquark condensation and color superconductivity is a subject of many recent theoretical studies [1]-[15]. In particular, as first pointed out by Alford, Rajagopal and Wilczek [4], in the case of 3 light flavors the diquark condensate can "lock" color and flavor symmetry rotations. The result is the color-flavor-locking (CFL) phase where an interesting pattern of chiral symmetry breaking emerges: $SU(3)_L \times SU(3)_R \times SU(3)_{\rm color} \to SU(3)_{L+R+{\rm color}}$. The CFL phase has many similarities with the chiral symmetry breaking QCD vacuum and nuclear matter. This observation has lead to the conjecture that quark and nuclear matter might be continuously connected [5].

As already pointed out in Refs. [4, 5], the CFL phase is characterized by a hierarchy of energy scales. At the lowest scale lie 10 (pseudo-)Goldstone modes, arising from the breaking of axial flavor symmetry, baryon $U(1)_B$ symmetry, and axial $U(1)_A$ symmetry. These modes, except for the $U(1)_A$, would be massless if quark masses were zero, but in reality they do have small masses. At a higher scale there are quark excitations, which are separated by the superconducting (BCS) gap Δ, and the gluons which acquire mass of order $g\mu$ from the Meissner effect. The Goldstone modes, therefore, dominate the physics at energy scales smaller than Δ and can be described by an effective theory. When quarks are massless, the Lagrangian of such a theory can be shown to be just the QCD chiral Lagrangian, which contains the decay constants and the velocities of the Goldstone bosons as free parameters [12, 13].

In this paper, we show that *all* parameters of the chiral Lagrangian, including the mass term, can be completely determined in the weak-coupling regime, or the regime of very high densities. Our results can be summarized as follows. We will use the "vacuum" notations for the Goldstone bosons, where we have an octet of pseudoscalars π, K and η and a singlet pseudoscalar η'. The Goldstone boson related to the breaking of $U(1)_B$ will be denoted as H. To the leading order in strong coupling, the decay constants of particles in the octet and of the singlets η' and H are linear in the chemical potential, μ:

$$f_\pi \approx 0.209\mu \qquad \text{and} \qquad f_H = f_{\eta'} \approx 0.195\mu. \tag{1}$$

The dispersion relation of the mesons has the form $\epsilon^2 = v^2p^2 + m^2$, where the maximum velocities of all the mesons are close to the speed of sound in ultrarelativistic fluids:

$$v_\pi = v_H = v_{\eta'} = \frac{1}{\sqrt{3}}. \tag{2}$$

The H meson remains exactly massless, while all the others acquire masses when nonzero quark masses m_q are taken into account. Due to the approximate $U(1)_A$ symmetry at large

μ the meson masses are not proportional to $m_q^{1/2}$, but to m_q [4]. In the limit $m_u, m_d \ll m_s$, 5 mesons (η and the kaons) have masses of the order m_s, while 4 others (η' and the pions) have masses of the order $(m_{u,d} m_s)^{1/2}$. The masses of the charged pions and the kaons are given by:

$$
\begin{aligned}
m_{\pi^\pm}^2 &= C(m_u + m_d)m_s + 2C'(m_u + m_d)(2m_u + 2m_d + m_s);\\
m_{K^0}^2 &= C(m_d + m_s)m_u + 2C'(m_d + m_s)(2m_d + 2m_s + m_u);\\
m_{K^\pm}^2 &= C(m_u + m_s)m_d + 2C'(m_u + m_s)(2m_u + 2m_s + m_d);
\end{aligned} \tag{3}
$$

where $C \approx 1.578$, and $C' \approx 0.04216$. The states of π^0, η and η' mix and their mass matrix is complicated. In the limit $m_s \gg m_{u,d}$ the heaviest is the η meson, while the η' is the lightest.

As we shall see, the meson spectrum of the CFL phase, although similar to that of QCD vacuum, has a certain inverse mass ordering: in particular, the neutral kaons are lighter than the charged kaons, and the lightest particle is the η' singlet. The kaons would also be lighter than the pions if it was not for a seemingly small, but important, second term in (3), proportional to C'. We shall give a simple explanation of these facts below.

2 The effective Lagrangian

In this section we review the basic arguments of Ref. [13]. The ground state of the CFL phase is characterized by the following diquark condensates[1]:

$$
X^{ia} \sim \epsilon^{ijk}\epsilon^{abc}\langle \psi_L^{bj}\psi_L^{ck}\rangle^* \qquad \text{and} \qquad Y^{ia} \sim \epsilon^{ijk}\epsilon^{abc}\langle \psi_R^{bj}\psi_R^{ck}\rangle^*, \tag{4}
$$

where the complex conjugation was added for convenience so that X and Y transform under $SU(3)_c \times SU(3)_L \times SU(3)_R$ as (3,3,1) and (3,1,3) respectively:

$$
X \to U_L X U_c^T \text{ and } Y \to U_R Y U_c^T. \tag{5}
$$

The low-energy excitations in the CFL phase are given by the slow rotations of the phases of X and Y. Therefore, we can factor out the norm of the condensates and consider unitary matrices X and Y. Together they give us $9 + 9 = 18$ degrees of freedom. 8 of them are eaten by the gluons through the Higgs mechanism, and the surviving 10 become low-energy excitations.

One of the Goldstones corresponds to the spontaneous breaking of the $U(1)_A$ symmetry, which, being a symmetry of the Lagrangian, is violated at the quantum level by the axial anomaly, or, equivalently, by instanton-induced interactions. This violation, however, becomes very small at large μ due to the effect of screening which suppresses the instanton

[1]There is also an admixture of 6-plet condensates, which, however, is small [9].

density by a high power of $1/\mu$ [10]. Therefore, we can treat the $U(1)_A$ on equal footing with other global symmetries in this regime.

For simplicity, we shall at first ignore the overall $U(1)$ phases of X and Y and assume, as in Ref. [13], $\det X = \det Y = 1$. The Lagrangian should be symmetric under the $SU(3)_c \times SU(3)_L \times SU(3)_R$ rotations (5). This condition fixes the Lagrangian to the leading (second) order in derivatives:

$$\mathcal{L}_{\text{eff}} = \frac{f_\pi^2}{2} \text{Tr} \left[(X^\dagger \partial_0 X)^2 + (Y^\dagger \partial_0 Y)^2 \right] + \text{spatial gradients} \tag{6}$$

The spatial gradients enter in a similar way but with a different constant instead of f_π. The cross term $\text{Tr}(X^\dagger \partial_0 X)(Y^\dagger \partial_0 Y)$ [13] is allowed by the symmetries, but, as we shall see in Sec. 3, it is suppressed in weak coupling (i.e., at large μ.) Roughly speaking, to leading order in g^2, left and right quarks decouple from each other.

Since $SU(3)_c$ is a local gauge symmetry we must replace the derivatives in Eq. (6) by covariant derivatives,

$$\begin{aligned} \mathcal{L}_{\text{eff}} &= \frac{f_\pi^2}{2} \text{Tr} \left[(X \partial_0 X^\dagger - g A_0)^2 + (Y \partial_0 Y^\dagger - g A_0)^2 \right] + \ldots \\ &= \frac{f_\pi^2}{4} \text{Tr} \left[(X \partial_0 X^\dagger - Y \partial_0 Y^\dagger)^2 + (X \partial_0 X^\dagger + Y \partial_0 Y^\dagger - 2g A_0)^2 \right] + \ldots \end{aligned} \tag{7}$$

The second term is responsible for the Higgs effect: the vector-like fluctuations, $dX = dY$, of $SU(3)$ phases become longitudinal components of the gluon A_μ. The gluons acquire a mass of order $O(gf_\pi)$ (electric and magnetic masses are, in general, different). We shall see that $f_\pi \sim \mu$, and thus the gluon mass is much larger than the momentum scales p that we are considering ($p < 2\Delta \ll g\mu$), so gluons decouple from the low-energy theory. The axial-like fluctuations of the phases, $dX = -dY$, can be written as fluctuations of the phases of a new unitary matrix:

$$\Sigma = XY^\dagger, \tag{8}$$

and the effective Lagrangian takes the form (in Euclidean space):

$$\mathcal{L}_{\text{eff}} = \frac{f_\pi^2}{4} \text{Tr} \left[\partial_0 \Sigma \partial_0 \Sigma^\dagger + v_\pi^2 \partial_i \Sigma \partial_i \Sigma^\dagger \right]. \tag{9}$$

which is the usual Lagrangian of the nonlinear sigma model except that the speed of the mesons, v_π, can be different from the speed of light. The matrix Σ is a singlet in color and transforms under $SU(3)_L \times SU(3)_R$ as

$$\Sigma \to U_L \Sigma U_R^\dagger, \tag{10}$$

and describes the meson octet.

What happens if we take into account the $U(1)$ phases of X and Y? Then it is possible to add into the chiral Lagrangian a term proportional to

$$(\mathrm{Tr}X^\dagger\partial_0 X)^2 + (\mathrm{Tr}Y^\dagger\partial_0 Y)^2. \tag{11}$$

It is possible to add a cross term, $(\mathrm{Tr}X^\dagger\partial_0 X)(\mathrm{Tr}Y^\dagger\partial_0 Y)$ but it is suppressed in weak coupling by the same reason by which the cross term was omitted in Eq. (6). We shall make the consequences of the term (11) clear by splitting X and Y into $SU(3)$ and $U(1)$ parts,

$$X = \tilde{X}e^{2i\theta+2i\phi} \quad \text{and} \quad Y = \tilde{Y}e^{-2i\theta+2i\phi}; \tag{12}$$

where $\tilde{X}$, $\tilde{Y}$ are $SU(3)$ matrices, and the angle ϕ is the variable conjugate to the baryon charge, normalized to 1 for a single quark. The normalization of the $U_A(1)$ phase θ is fixed analogously. Consequently, the field Σ defined in Eq. (8) now has the form

$$\Sigma = \tilde{\Sigma}\; e^{4i\theta}, \qquad \tilde{\Sigma} = \tilde{X}\tilde{Y}^\dagger\;. \tag{13}$$

In terms of $\tilde{\Sigma}$, ϕ and θ, the lowest order chiral Lagrangian, consistent with the symmetries, has the form

$$\begin{aligned} \mathcal{L}_{\text{eff}} \;&=\; \frac{f_\pi^2}{4}\mathrm{Tr}\left[\partial_0\tilde{\Sigma}\partial_0\tilde{\Sigma}^\dagger + v_\pi^2\partial_i\tilde{\Sigma}\partial_i\tilde{\Sigma}^\dagger\right] + 12 f_{\eta'}^2\left[(\partial_0\theta)^2 + v_{\eta'}^2(\partial_i\theta)^2\right] \\ &\quad + 12 f_H^2\left[(\partial_0\phi)^2 + v_H^2(\partial_i\phi)^2\right]. \end{aligned} \tag{14}$$

The difference between the decay constants of η' and H mesons from f_π arises from the additional allowed term (11), while the difference between $f_{\eta'}$ and f_H arises from the cross term $(\mathrm{Tr}X^\dagger\partial_0 X)(\mathrm{Tr}Y^\dagger\partial_0 Y)$, and, therefore, is small at weak coupling. We shall use $f_{\eta'} = f_H$ in the rest of the paper.

The elementary meson fields π^A, η', and H, are defined as

$$\tilde{\Sigma} = \exp\left(i\frac{\lambda^A\pi^A}{f_\pi}\right), \qquad \theta = \frac{\eta'}{\sqrt{24}f_{\eta'}}, \qquad \phi = \frac{H}{\sqrt{24}f_H}, \tag{15}$$

where λ^A $(A = 1\ldots 8)$ are Gell-Mann matrices normalized so that $\mathrm{Tr}\lambda^A\lambda^B = 2\delta^{AB}$.

3 Decay constants of the Goldstone bosons

Let us now show that all the decay constants of the Goldstone bosons, f_π, $f_{\eta'}$, and f_H, can be computed in the high-density regime and are all proportional to the chemical potential. We will demonstrate our method on f_H, whose calculation is the simplest. Let us imagine that baryon symmetry is not a global symmetry, but instead a local one. So we introduce into the theory a gauge field A_μ coupled to the baryon current. Due to the Higgs mechanism,

the gauge bosons acquire a finite mass, which can be computed in both the microscopic theory, where it is expressed via the chemical potential, and in the effective theory, where it is proportional to f_H. Comparison of the two results leads to the determination of f_H.

In order to compute f_H we have to deal only with the part of the effective chiral Lagrangian (14) that contains the phase ϕ. Since ϕ is not neutral with respect to the baryon symmetry, which is now local, one should replace the derivatives by covariant ones,

$$\mathcal{L} = 12 f_H^2 \left[(\partial_0 \phi + e A_0)^2 + v_H^2 (\partial_i \phi + e A_i)^2 \right], \tag{16}$$

where e is some arbitrary small coupling constant. From Eq. (16) we find that the gauge bosons acquire finite masses, which are different for A_0 and A_i,

$$m_{A_0}^2 = 24 e^2 f_H^2 \tag{17}$$
$$m_{A_i}^2 = 24 v_H^2 e^2 f_H^2 \tag{18}$$

On the other hand, the masses in Eqs. (17,18) can be computed from the microscopic theory, where $m_{A_0}^2$ has the meaning of the Debye mass (or the inverse Thomas-Fermi screening length) of the electric A_μ field, while $m_{A_i}^2$ is the Meissner mass (or the inverse London penetration depth) of the magnetic components of A_μ. To compute these masses, it is most convenient to use a low-energy effective theory containing only fermion modes near the Fermi surface, similar to the one derived by Hong [14]. We shall work in Euclidean space where the action has the form

$$S = \int \frac{d^4p}{(2\pi)^4} \left(\psi^\dagger(p)(ip_0 + \epsilon_p)\psi(p) + e\psi^\dagger(p)\psi(p)(iA_0(0) + v_i A_i(0)) \right) + \int d^4x \, \frac{m_0^2}{2} A_i A_i \tag{19}$$

where $v_i = p_i/|\mathbf{p}|$, and we have kept only the gluon modes with zero momentum in the interaction term since only they are needed in future discussion. By using the effective Lagrangian (19) one can avoid dealing with the Dirac structure of the quark propagator, which is somewhat complicated in the superconducting phase. The "bare Meissner mass" term proportional to m_0^2 in Eq. (19) emerges from integration out of all degrees of freedom except the ones near the Fermi surface; its magnitude can be found from the following simple argument. In the first-quantization picture, the Hamiltonian of a relativistic particle is simply $\mathcal{H} = |\mathbf{p}|$. As we couple the particle to the gauge field A_μ, its Hamiltonian becomes

$$\mathcal{H} = |\mathbf{p} + e\mathbf{A}| + ieA_0 = |\mathbf{p}| + ieA_0 + e(\mathbf{v}\cdot\mathbf{A}) + \frac{e^2}{2|\mathbf{p}|} \left(\mathbf{A}^2 - (\mathbf{v}\cdot\mathbf{A})^2 \right) + \ldots \tag{20}$$

In the second quantization language, the first three terms in Eq. (20) correspond to the first term of the Lagrangian (19). The last term, if we sum over all particles with momentum $\mathbf{p}$

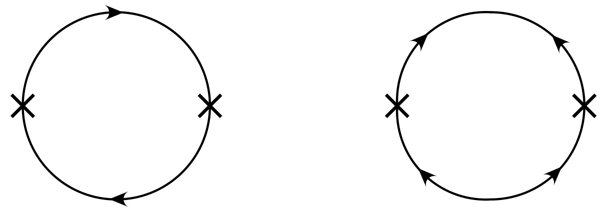

Figure 1: The leading order diagrams contributing to decay constants.

inside the Fermi sphere, reproduce the bare Meissner mass term in the effective Lagrangian (19) with

$$m_0^2 = 6e^2 \frac{\mu^2}{2\pi^2} \tag{21}$$

A consistency check for the Lagrangian (19) is the computation of the physical Meissner mass in the normal phase. It can be shown that the bare value (21) is exactly the one required for the Meissner effect to be absent in the normal phase.

In the CFL phase, the quark propagator has its simplest form in the basis

$$\psi_{ai} = \sum_{A=1}^{9} \frac{\lambda^A_{ai}}{\sqrt{2}} \psi^A \tag{22}$$

where λ^A are Gell-Mann matrices if $A = 1 \dots 8$ and $\lambda^9 = \sqrt{2/3}$. The Nambu-Gorkov quark propagators are diagonal in this basis,

$$\langle \psi^A(p)\psi^B(-p)\rangle = \frac{\delta^{AB}\Delta^A}{p_0^2 + \epsilon_p^2 + \Delta_A^2}; \tag{23}$$

$$\langle \psi^A(p)\psi^{\dagger B}(-p)\rangle = \frac{\delta^{AB}(ip_0 + \epsilon_p)}{p_0^2 + \epsilon_p^2 + \Delta_A^2}; \tag{24}$$

where $\epsilon_p = |\mathbf{p}| - \mu$ is the energy relative to the Fermi surface. The gaps Δ_A with $A = 1 \dots 8$ are equal, but different from Δ_9. In weak coupling, if we denote $\Delta_1 = \cdots = \Delta_8 = \Delta$, then $\Delta_9 = -2\Delta$.[2]

To the leading order, the only contribution to $m^2_{A_0}$ is from the sum of two one-loop diagrams as shown in Fig. (1), which, for zero external momentum, is equal to

$$2\frac{\mu^2}{2}\int \frac{dp_0\, d\epsilon_p}{(2\pi)} \sum_{A=1}^{9}\left(-\frac{(ip_0+\epsilon_p)^2}{(p_0^2+\epsilon_p^2+\Delta_A^2)^2} + \frac{\Delta_A^2}{(p_0^2+\epsilon_p^2+\Delta_A^2)^2}\right), \tag{25}$$

where we use the formula for the phase space near the Fermi surface, $\int \frac{d^4p}{(2\pi)^4} = \frac{\mu^2}{2\pi^2}\int \frac{dp_0}{2\pi}\, d\epsilon_p$. The overall factor of 2 in Eq. (25) comes from summation over left- and right-handed quarks

[2] Our Δ^A with $A \neq 9$ and Δ^9 corresponds to Δ_8 and Δ_1 in the notation of Ref. [4].

in the internal loop. While the second diagram in Fig. (1) is finite, the first, treated formally, has logarithmic divergence. The prescription to deal with the divergence is known [16], and is essentially to perform the integration over p_0 first. We find that the two diagrams in Fig. (1) give equal contribution to the squared Debye mass which is given by

$$m_{A_0}^2 = 18e^2 \frac{\mu^2}{2\pi^2}. \tag{26}$$

This value is the same as the Debye mass that A_μ would have in the absence of the superconductivity. The origin of this coincidence can be made clear by the following argument. What we have computed is simply the 00 component of the polarization operator of A_μ, Π_{00}. Since A_0 is coupled to the baryon charge $n = \psi^\dagger\psi$, Π_{00} has the following interpretation in the linear response theory: the baryon charge density generated by an external uniform field A_0 is given by

$$n = \Pi_{00}(0)A_0. \tag{27}$$

But the uniform A_0 field is simply a shift of the chemical potential, or the Fermi energy. Therefore, Π_{00} is equal to $e^2\,\partial n/\partial\mu$, i.e., the density of states near the Fermi surface, which is exactly the right hand side of Eq. (26). It is easy to see that $\partial n/\partial\mu$ is the same in the normal phase and superconducting phase, provided that the gap in the latter is small.

Comparing Eq. (26) with what one gets from the effective theory, Eq. (17), one finds the decay constant f_H,

$$f_H^2 = \frac{3}{4}\frac{\mu^2}{2\pi^2}. \tag{28}$$

An important remark is in order here. Eq. (28) tells us that f_H depends only on the chemical potential μ, but not on the gap Δ. This might seem contradicting the fact that at $\Delta = 0$ the $U(1)_B$ symmetry would be restored and no Goldstone boson is expected in the theory. The explanation is that, as Δ decreases, the domain of applicability of the effective Lagrangian $p < 2\Delta$ shrinks and disappears at $\Delta = 0$; therefore the persistence of f_H does not contradict the restoration of $U(1)_B$ symmetry at $\Delta = 0$.

To find the Meissner mass, we have to evaluate the same two diagrams of Fig. (1), where the vertices are attached to A_i and A_j. Instead of being equal, the two diagrams now cancel each other, since the vertex factor of the first diagram is v_iv_j, while in the second diagram it is $v_i(-v_j)$. Therefore, the Meissner mass in the superconducting phase is equal to the bare Meissner mass in the effective Lagrangian (19). Compared to Eq. (18) and taking into account Eq. (28), one finds that the velocity of the H boson is equal to $1/\sqrt{3}$, i.e. the speed of sound in relativistic fluids,

$$v_H^2 = \frac{1}{3}. \tag{29}$$

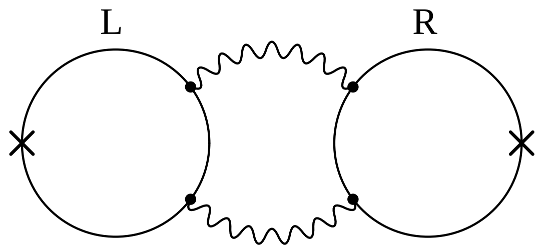

Figure 2: A higher-order diagram contributing to the decay constants. One loop contains left-handed quark, the other has right-handed quark.

This result is not completely surprising, since the $U(1)_B$ phase ϕ of the condensate is the variable conjugate to the baryon density $\psi^\dagger\psi$, whose fluctuations give rise to the sound. Therefore, H quanta can be considered as phonons. But they are not hydrodynamic phonons since they exist outside the hydrodynamic regime (as in our case, at zero temperature where the mean free path diverges.) Moreover, as we shall explain, if one increases the temperature the velocity of the Goldstone bosons decreases and becomes zero at the critical temperature, while the speed of the hydrodynamic sound is almost insensitive to the temperature in this range. Therefore, the fact that the velocity of H quanta (and other Goldstone bosons, as we shall see) at zero temperature is equal to the speed of hydrodynamic sound should be considered as a coincidence bearing no fundamental reason.

For the η' meson, one can repeat the same calculation and see that, to the leading order, all diagrams remain the same. Thus we find that $f_{\eta'} = f_H$, and that the η' meson also propagates with the sound speed. There is, however, no symmetry that requires the decay constants of H and η' to be equal; in fact, they are not equal in higher orders of perturbation theory. For example, the digram drawn in Fig. 2 gives contribution of opposite signs to f_H and $f_{\eta'}$.

The computation of f_π and v_π is completely analogous to that of f_H. We could introduce a fictitious field coupled to the flavor currents, but we can also make use of the existing coupling to the gluons to compute f_π. The only additional complication is that the interaction vertex now has a non-trivial structure. In our basis (22), the interaction vertex has the form

$$\frac{1}{4}\sum_{A,C=1}^{9}\sum_{B=1}^{8}\mathrm{tr}(\lambda^A\lambda^B\lambda^C)\psi^{\dagger A}A^B\psi^C \tag{30}$$

Straightforward calculations show that the first diagram in Fig. 1 gives $(3/4)g^2\mu^2/(2\pi^2)$ and the second diagram contributes $-((3+4\ln 2)/18)g^2\mu^2/(2\pi^2)$ to the Debye mass. Taking into

account the factor of 2 from left- and right-handed quarks, the result reads

$$m_{A_0}^2 = \frac{21 - 8\ln 2}{18}\frac{g^2\mu^2}{2\pi^2} \tag{31}$$

The mass in Eq. (31) is not equal to the Debye mass in the normal phase, in contrast to the previous case of the fictitious $U(1)$ boson coupled to the baryon current.

Notice that if the gluons are coupled to the left-handed quarks alone, the squared Debye mass would be half smaller, which is exactly what one obtains by throwing away Y from the Lagrangian (7). However, if we add the XY cross term to the Lagrangian, this will no longer be true. Therefore, this cross term, though not forbidden by the symmetry, has a small coefficient in weak coupling. This can be seen also from the fact that the coupling of the left- and right-handed flavor currents is zero at the leading order. In higher orders of perturbation theory it receives contribution from the diagram like in Fig.2, where one vertex corresponds to the left-handed flavor current and the other to the right-handed one.

In the effective theory, the gluon mass is given by $m_{A_0}^2 = g^2 f_\pi^2$. Therefore, one can determine the decay constants of the mesons in the pseudoscalar octet,

$$f_\pi^2 = \frac{21 - 8\ln 2}{18}\frac{\mu^2}{2\pi^2}. \tag{32}$$

The ratio $f_\pi^2/f_{\eta'}^2 = f_\pi^2/f_H^2 = 2(21 - 8\ln 2)/27 \approx 1.14$ is quite close to one. This fact seems to stem from the OZI rule [17].

The Meissner mass is equal to

$$m_{A_i}^2 = 2\frac{\mu^2}{2\pi^2}\left(\frac{1}{2} - \frac{1}{4} - \frac{3 + 4\ln 2}{54}\right) = \frac{21 - 8\ln 2}{54}\frac{g^2\mu^2}{2\pi^2}, \tag{33}$$

where, in the parenthesis in the intermediate expression, the first term comes from the bare Meissner mass, while the two last terms come from the two diagrams in Fig. 1. Again, in contrast with the case of $U(1)_B$, the contribution of the two diagrams does not cancel each other. It is somewhat surprising that the squared Meissner mass is 1/3 of the squared Debye mass, as it is for the $U(1)_B$ case. Since the ratio of these two masses is the velocity of the Goldstone bosons, we find that the velocities of *all* Goldstone modes in our theory is equal to the speed of sound $1/\sqrt{3}$ to the leading order of perturbation theory.[3]

4 Meson masses

We now turn on finite small quark masses and compute the resulting masses of the Goldstone bosons. By introducing bare quark masses into the microscopic theory, we break the

[3]Similar results, $v = v_F/\sqrt{3}$, have been found by Bogolyubov and Anderson for phase waves in BCS superconductors and by Leggett for spin waves in superfluid ^{3}He [18].

$SU(3)_L \times SU(3)_R$ symmetry. This means that more terms are allowed in $\mathcal{L}_{\text{eff}}$. However, constraints on the form of these terms can still be imposed if we note that the bare quark mass term

$$\Delta\mathcal{L} = \psi_L^\dagger M \psi_R + \text{h.c.}, \tag{34}$$

with M being the 3×3 mass matrix, could be made invariant under the $SU(3)_L \times SU(3)_R$ if the matrix M is not passive under this symmetry but also transforms together with $\psi_{L,R}$ in the following way,

$$\psi_L \to U_L \psi_L, \quad \psi_R \to U_R \psi_R \quad \text{and} \quad M \to U_L M U_R^\dagger. \tag{35}$$

Any term in the effective Lagrangian, written as a function of Σ and M, must respect the extended symmetry (10), (35). At large μ the $U(1)_A$ symmetry is effectively restored, which imposes an additional constraint on the possible form of mass terms in the chiral Lagrangian. Under the $U(1)_A$ transformation, the microscopic degrees of freedom transform as

$$\psi_L \to e^{i\zeta}\psi_L, \quad \psi_R \to e^{-i\zeta}\psi_R \tag{36}$$

where ζ is an arbitrary pure phase. Eq. (36) implies the following transformation law of X, Y and Σ,

$$X \to e^{-2i\zeta} X, \quad Y \to e^{2i\zeta} Y \quad \text{and} \quad \Sigma \to e^{-4i\zeta}\Sigma, \tag{37}$$

The quark mass term (34) is invariant under the $U(1)_A$ symmetry if one requires that M transforms as

$$M \to e^{2i\zeta} M. \tag{38}$$

under the $U(1)_A$. Thus, any mass term in the effective Lagrangian must be invariant under both the $SU(3)_L \times SU(3)_R$ and the $U(1)_A$ symmetry extended by the transformation of M.[4]

As the bare quark masses are assumed small, we want to construct the mass terms of lowest possible order in M. To the first order, one candidate is the mass term of the chiral Lagrangian at zero chemical potential, $\text{Tr}M^\dagger\Sigma$. This term is invariant under the $SU(3)_L \times SU(3)_R$, however it is not invariant under the $U(1)_A$ symmetry. Therefore, we must consider terms of higher order in M.[5]

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[4] A similar symmetry argument allows one to fix the form of the effective Lagrangian in 2-color QCD to lowest order in μ and m_q [11].

[5] The fact that $\Delta\mathcal{L}_{\text{eff}} = O(M^2)$ means that in the chiral limit $M \to 0$ the chiral condensate $\langle\bar\psi\psi\rangle = 0$, while $\langle(\bar\psi\psi)^2\rangle \neq 0$. Such a pattern of spontaneous chiral symmetry breaking has been discussed in [19].

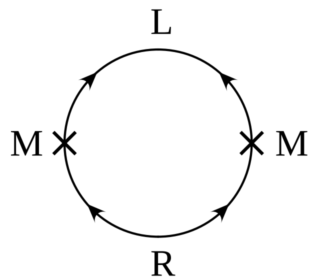

Figure 3: The leading order diagram contributing to the shift of the vacuum energy by a small quark mass M.

In the order M^2 we find that the following term is allowed by the symmetry[6]

$$\Delta\mathcal{L}_{\text{eff}} = -c \cdot \det M \cdot \mathrm{Tr}(M^{-1}\Sigma) + \text{h.c.} \tag{39}$$

Another term allowed by the symmetry is[7]

$$\Delta\mathcal{L}'_{\text{eff}} = -c' \left[\mathrm{Tr}(M\Sigma^\dagger)^2 + (\mathrm{Tr}M\Sigma^\dagger)^2\right] \det\Sigma + \text{h.c.} \tag{40}$$

The coefficients c and c' of the mass terms from (39) and (40) can be calculated by matching the shift of the vacuum energy as a function of Σ they induce in the effective theory: $\Delta\mathcal{E}_{\text{vac}}(\Sigma) = \Delta\mathcal{L}_{\text{eff}} + \Delta\mathcal{L}'_{\text{eff}}$, to the perturbative calculation in the microscopic theory. The nontrivial, Σ-dependent, shift in the vacuum energy due to (34) is given to the lowest order by the diagram in Fig.3, which is similar to the second diagram in Fig. 1. Another diagram, similar to the first diagram in Fig 1, also contributes to the shift of the vacuum energy, but its contribution does not depend on Σ.[8] As a result, terms such as $\mathrm{Tr}(M\Sigma^\dagger M^\dagger\Sigma)$ and $(\mathrm{Tr}M\Sigma^\dagger)(\mathrm{Tr}M^\dagger\Sigma)$, in principle, allowed by the symmetry, are absent to the order we are working.

The calculation of the diagram in Fig.3 gives:

$$c = \frac{51 + 32\ln 2}{108}\frac{\mu^2}{2\pi^2} \qquad \text{and} \qquad c' = \frac{15 - 16\ln 2}{216}\frac{\mu^2}{2\pi^2}. \tag{41}$$

[6] One way of looking at this term is to realize that under $SU(3)_L$ both M and Σ transform as fundamental 3-plets. We require two powers of M and one of Σ in order to satisfy the $U(1)_A$ neutrality. We can construct an $SU(3)_L$ singlet out of a product of M, M and Σ if we antisymmetrize with respect to the first index of each of these matrices (the index on which $SU(3)_L$ acts). Antisymmetrizing also with respect to the second index we obtain an $SU(3)_L \times SU(3)_R$ singlet: $\epsilon_{a'b'c'}\epsilon_{abc}M_{aa'}M_{bb'}\Sigma_{cc'}$, which coincides with (39) up to a constant.

[7] The alternative linear combination of the two terms in (40) is equivalent to the term (39): $\left[\mathrm{Tr}(M\Sigma^\dagger)^2 - (\mathrm{Tr}M\Sigma^\dagger)^2\right]\det\Sigma = 2\det M \cdot \mathrm{Tr}(M^{-1}\Sigma)$.

[8] This is related to the fact that such a diagram does not depend on the values of the gaps Δ on its two fermion lines, even if they are different.

The coefficient c' is numerically very small: $c'/c \approx 0.027$, and is related to the fact that one of the 9 gaps in the fermion propagator matrix (23) is different from the other 8. This completely determines the mass term in the effective Lagrangian given by (39) and (40).

To write the $(\text{mass})^2$ matrix for the Goldstone bosons we expand the field Σ:

$$\Sigma = \exp\left(\frac{i\pi^a\lambda^a}{f_\pi}\right) = 1 + \frac{i\pi^a\lambda^a}{f_\pi} - \frac{\pi^a\pi^b\lambda^a\lambda^b}{2f_\pi^2} + \dots\ , \tag{42}$$

where $a = 1, \dots 8, 9$, $\lambda^9 = \sqrt{2/3}$ and $\pi^9 = \eta' f_\pi / f_{\eta'}$. Since the kinetic term is conventionally normalized we can read off the $(\text{mass})^2$ matrix, $\mathcal{M}^2_{ab} + \mathcal{M}'^2_{ab}$, from the mass terms (39), (40) (except for the trivial rescaling of the η' field):

$$\begin{aligned}
\frac{1}{2}\mathcal{M}^2_{ab}\pi^a\pi^b &= \frac{C}{2}\det M \cdot \mathrm{Tr}(M^{-1}\lambda^a\lambda^b)\pi^a\pi^b;\\
\frac{1}{2}\mathcal{M}'^2_{ab}\pi^a\pi^b &= C'\Big\{\pi^a\pi^b\left[(\mathrm{Tr}M\lambda_a)(\mathrm{Tr}M\lambda_b) + (\mathrm{Tr}M)(\mathrm{Tr}M\lambda_a\lambda_b)\right.\\
&\quad \left.+(\mathrm{Tr}M\lambda_a M\lambda_b) + (\mathrm{Tr}M^2\lambda_a\lambda_b)\right] + 3(\pi^9)^2\left[(\mathrm{Tr}M)^2 + (\mathrm{Tr}M^2)\right]\\
&\quad -2\sqrt{6}\,\pi^a\pi^9\left[(\mathrm{Tr}M)(\mathrm{Tr}M\lambda_a) + (\mathrm{Tr}M^2\lambda_a)\right]\Big\};
\end{aligned} \tag{43}$$

where, using (32) and (41),

$$C = \frac{2c}{f_\pi^2} = \frac{1}{3}\frac{51 + 32\ln 2}{21 - 8\ln 2} \approx 1.578 \quad \text{and} \quad C' = \frac{2c'}{f_\pi^2} = \frac{1}{6}\frac{15 - 16\ln 2}{21 - 8\ln 2} \approx 0.04216. \tag{44}$$

Since these are pure numbers, the meson masses depend only on the bare quark masses, in contrast to the QCD vacuum where they also depend on the value of the chiral condensate.

In order to understand the pattern of masses let us begin with the first term in (43), since the coefficient of this term, C, is much greater than that of the second term, C'.

With the quark mass matrix $M = \mathrm{diag}(m_u, m_d, m_s)$, the 9×9 meson mass matrix (43) decomposes into a diagonal 6×6 matrix and a non-diagonal 3×3 matrix. The former gives rise to the mass formulas for $\pi^\pm$, $K^\pm$, K^0 and $\bar{K}^0$:

$$\begin{aligned}
m^2_{\pi^\pm} = C(m_u + m_d)m_s + \dots; \quad m^2_{K^\pm} &= C(m_u + m_s)m_d + \dots;\\
m^2_{K^0} &= C(m_d + m_s)m_u + \dots\ ,
\end{aligned} \tag{45}$$

where ellipses denote contributions proportional to C'. The remaining 3×3 matrix corresponding to the π^0, η and η' mesons is not diagonal. The mixing pattern is easy to understand if we neglect the small difference between f_π and $f_{\eta'}$. Then the 3×3 mass matrix has the simplest form in the basis $\bar{u}u$, $\bar{d}d$, $\bar{s}s$, related to π^0, η, η' by the quark-model relations $\pi^0 = (\bar{u}u - \bar{d}d)/\sqrt{2}$, $\eta = (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}$, and $\eta' = (\bar{u}u + \bar{d}d + \bar{s}s)/\sqrt{3}$. In this basis $\lambda_{\bar{u}u} = \mathrm{diag}(1, 0, 0)$, $\lambda_{\bar{d}d} = \mathrm{diag}(0, 1, 0)$, $\lambda_{\bar{s}s} = \mathrm{diag}(0, 0, 1)$, and the 3×3 mass matrix

becomes diagonal. The mixing between π^0, η, and η', thus, is ideal, and the masses of the mixed states are

$$m^2_{\bar{u}u} \cong 2Cm_d m_s + \ldots, \qquad m^2_{\bar{d}d} \cong 2Cm_u m_s + \ldots, \qquad m^2_{\bar{s}s} \cong 2Cm_u m_d + \ldots. \tag{46}$$

As we shall see, however, the second mass term makes a significant correction to this ideal mixing, despite the fact that $C' \ll C$, when $m_s \gg m_{u,d}$.

The reader will notice that the mass ordering induced by the first term in (43) is completely reversed compared to what one sees in the QCD vacuum. While this is quite surprising, and seems unnatural under the hypothesis of continuity between quark and hadronic matter, it can be easily explained. The key point is that the mesons, which are the fluctuations of Σ in Eq. (8), should be thought of as bound states of a triplet antidiquark X and an antitriplet diquark $Y^\dagger$, i.e. as $\bar{q}\bar{q}qq$ states, rather than $\bar{q}q$ states. Consider, for example, the $\bar{s}s$ state. Now the s quark should be replaced by the $\bar{u}\bar{d}$ antidiquark, and the $\bar{s}$ antiquark should be replaced by the ud diquark, so this meson is represented as $\bar{u}\bar{d}ud$. Since such a meson does not contain the strange quark, it is not surprising that its mass does not depend on m_s. For all other mesons, one can write the quark structure as well, by making the replacement $u \to \bar{d}\bar{s}$, $d \to \bar{u}\bar{s}$, $s \to \bar{u}\bar{d}$. Since one replaces the heaviest quark by the lightest antidiquark and vice versa, the inverse mass ordering can be expected.[9] It is important to note, however, that the mesons in the CFL phase have the same quantum numbers (up to mixing) as the mesons in the QCD vacuum.

Although the coefficient of the second term in (43), C', is almost 40 times smaller than C, it contributes significantly to the masses of some of the mesons, because the ratio of m_s to m_d or m_u is also very large. The masses of the π and K mesons are given by (3). In the limit $m_s \gg m_{u,d}$ they reduce to:

$$m^2_{\pi^\pm} \approx C(m_u + m_d)m_s; \quad m^2_{K^0} \approx (Cm_u + 4C'm_s)m_s; \quad m^2_{K^\pm} \approx (Cm_d + 4C'm_s)m_s; \tag{47}$$

where we treated both C/C' and m_s/m_d as large numbers of the same order of magnitude. We see that in the kaon masses the smallness of C'/C is compensated by the greatness of $m_s/m_{u,d}$. The ordering of charged and neutral kaons remains inverted, however.

The remaining 3×3 matrix for π_0, η and η' is rather complicated. To understand its properties it is helpful to consider the limiting case when $m_u = m_d = 0$. In this case the matrix simplifies, and one finds that two of the three states are massless (together with charged pions, in accordance with the Goldstone theorem and the breaking of $SU(2) \times U(1)_A$), while the third has a mass of order m_s. The massless states are pure π^0 and $(2\sqrt{2}\pi^9 - \pi^8)/3$, which is mostly η'.

[9] It has been suggested that a similar pattern arises for scalar mesons in QCD vacuum [20].

In order to get the feeling of the numbers involved, we substitute for the bare quark masses the values $m_u = 4$ MeV, $m_d = 7$ MeV and $m_s = 150$ MeV. Performing numerical diagonalization of the mass matrix (43) we find the following spectrum: $m_\eta = 117$ MeV, $m_{\pi^0} = 53$ MeV, $m_{\eta'} = 30$ MeV. The mixings of π^0 with η and η' are on the level of a few percent, while the mixing between η and η' is on the level of 20%. The numerical values for the masses (3) of pions and kaons are given by: $m_{\pi^\pm} = 53$ MeV, $m_{K^0} = 76$ MeV, $m_{K^\pm} = 72$ MeV.

5 Conclusion

In this paper we have shown that all parameters characterizing the dynamics of Goldstone bosons in the CFL phase, including the decay constants, masses, and the maximum velocities, can be reliably calculated in the weak-coupling regime of high densities. These parameters depend only on very few inputs — in fact, only the decay constants depend on the chemical potential, while the meson masses depend only on the bare quark masses and the velocities are given by a constant. This fact allows us to derive the full chiral Lagrangian without being dependent of the calculation of, e.g., the BCS gap, whose asymptotic behavior is known [6], but the exact numerical value of the prefactor is not [7, 8].

In contrast to the meson masses in the QCD vacuum, we found that, when the strange quark is much heavier than the u and d quarks, 5 mesons have masses of the order of m_s, and 4 mesons have masses of the order $(m_{u,d}m_s)^{1/2}$. The lightest of those mesons is η' with a mass around 30 MeV. The most surprising fact we found is the partially inverse mass ordering of the meson nonet. As we explained, this can be understood if we treat the mesons as $\bar{q}\bar{q}qq$ states, rather than $\bar{q}q$ states.

We must also discuss the region of validity of our treatment. The effective theory we are working with contains only meson modes, therefore, it is applicable only for energies smaller than 2Δ. In particular, a meson exists only as long as its mass is smaller than 2Δ, otherwise it would rapidly decay into a particle-hole pair. At asymptotically large chemical potential where Δ increases with μ, all mesons are stable with respect to this decay channel. At intermediate densities, the gap may actually drop substantially below 100 MeV in some range of the chemical potential [7]. If the gap is small, some of the mesons may become unstable. For example, for our values of the quark masses, the η meson disappears if Δ is smaller than about 60 MeV, and the lightest meson, η', disappears only when the gap drops below 15 MeV.

Another effect which becomes important when μ is not sufficiently large is the explicit breaking of the $U(1)_A$ symmetry by anomalous instanton-induced interactions. This effect gives a direct contribution (independent of the quark masses) to the mass of the η' state,

and $m_q^{1/2}$ contributions to the masses of the other mesons [4].

We can estimate the value of μ at which the transition from the normal QCD vacuum to the CFL phase occurs very roughly by comparing the value of $f_\pi \approx 0.2\mu$ in the CFL phase to its vacuum value $f_\pi = 93$ MeV. We find $\mu_{\rm tr} \sim 500$ MeV, which is reasonable and agrees with other estimates.

It is straightforward to extend our calculations to the case of finite temperature. Let us discuss, qualitatively, the case of temperatures close to critical. The Meissner masses should decrease as the temperature increases and vanish at the critical temperature, since one expects the Meissner effect to be absent in the normal phase. On the other hand, Debye masses are nonzero in the normal phase, and so should not vanish at the critical temperature. Thus, near the phase transition, the velocities of the Goldstone modes become small, which is the manifestation of the critical slowing down of the relaxation of the condensate near the phase transition. The decay constants remain finite, while the masses go to zero as $\mu\Delta/T_c$ near the phase transition. It would be interesting to study possible consequences of this behavior. The inverse mass ordering and anomalous lightness of mesons, in particular, of the η' [21], in the CFL phase may also significantly affect their production should such phase be accessible in heavy ion collisions.

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